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Institute for Logic, Language and Computation

Completeness and the FMP for KA, revisited

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- I learned most constructions as an undergraduate, here in Leiden.

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- Laws of Kleene algebra (KA) model equivalence of regular expressions.

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- When is something true *only by the laws of KA*?
- How can we concisely show that something is *not* provable in KA?

Kleene algebra

Definition

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Definition (Kleene algebra; c.f. Kozen 1994)

A *Kleene algebra* is a tuple $(K, +, \cdot, ^*, 0, 1)$ where

(1) The "usual" laws for $+$ and $\cdot$ hold (associativity, distributivity, etc. . .)

(2) For all $x, y, z \in K$, the following are true:

$$x + x = x \qquad 1 + x \cdot x^* = x^* \qquad 1 + x^* \cdot x = x^*$$

$$\frac{x + y \cdot z \leq z}{y^* \cdot x \leq z} \qquad \frac{x + y \cdot z \leq y}{x \cdot z^* \leq y}$$

Here, $x \leq y$ is a shorthand for $x + y = y$.

Kleene algebra

Languages

Fix a (finite) set of *letters* Σ, and write Σ^* for the set of words over Σ.

Example (KA of languages)

The KA of *languages over* Σ is given by $(\mathcal{P}(\Sigma^*), \cup, \cdot, ^*, \emptyset, \{\epsilon\})$, where

- $\mathcal{P}(\Sigma^*)$ is the set of sets of words (*languages*);
- $\cdot$ is pointwise concatenation, i.e., $L \cdot K = \{wx : w \in L, x \in K\}$;
- * is the Kleene star, i.e., $L^* = \{w_1 \cdots w_n : w_1, \ldots, w_n \in L\}$;
- ϵ is the empty word.

Fix a (not necessarily finite) set of *states* S.

Example (KA of relations)

The KA of *relations over* S is given by $(\mathcal{R}(S), \cup, \circ, ^*, \emptyset, \Delta)$, where

- $\mathcal{R}(S)$ is the set of relations on S;
- $\circ$ is relational composition.
- * is the reflexive-transitive closure.
- Δ is the identity relation.

Kleene algebra

Reasoning example

Claim

In every KA K and for all $u, v \in K$, it holds that $(u \cdot v)^* \cdot u \leq u \cdot (v \cdot u)^*$.

Kleene algebra

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Proof. First, let's recall the fixpoint rule:

$$\frac{x + y \cdot z \leq z}{y^* \cdot x \leq z}$$

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$$u + u \cdot v \cdot u \cdot (v \cdot u)^* = u \cdot (1 + v \cdot u \cdot (v \cdot u)^*)$$

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□

Kleene algebra

Expressions

Definition

Exp is the set of *regular expressions*, generated by

$$e, f ::= 0 \mid 1 \mid \mathtt{a} \in \Sigma \mid e + f \mid e \cdot f \mid e^*$$

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Given a KA $(K, +, \cdot, ^*, 0, 1)$ and $h : \Sigma \to K$, we define $\widehat{h} : \mathrm{Exp} \to K$ by

$$\widehat{h}(0) = 0 \qquad \widehat{h}(\mathtt{a}) = h(\mathtt{a}) \qquad \widehat{h}(e \cdot f) = \widehat{h}(e) \cdot \widehat{h}(f)$$

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Example

If $\ell : \Sigma \to \mathcal{P}(\Sigma^*)$ where $\ell(\mathtt{a}) = \{\mathtt{a}\}$, then $\widehat{\ell}(e)$ is the regular language denoted by e.

Kleene algebra

Model theory

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- $K, h \models e = f$ when K is a KA and $h : \Sigma \to K$ with $\widehat{h}(e) = \widehat{h}(f)$.

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- $\mathfrak{F} \models e = f$ when $K \models e = f$ holds in every *finite* KA K.

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- $\models e = f$ when $K \models e = f$ for every KA K.
- $\mathfrak{F} \models e = f$ when $K \models e = f$ holds in every *finite* KA K.
- $\mathfrak{R} \models e = f$ when $\mathcal{R}(S) \models e = f$ for all S.

Kleene algebra

Model theory

$$\models e = f$$

$$\Updownarrow \quad \text{(Kozen 1994)}$$

$$\mathcal{P}(\Sigma^*) \models e = f$$

Kleene algebra

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$$\mathfrak{R} \models e = f \overset{\text{(Pratt 1980)}}{\Longleftrightarrow} \mathcal{P}(\Sigma^*) \models e = f$$

Kleene algebra

Model theory

(Palka 2005)

$$\mathfrak{F} \models e = f \iff \models e = f$$

(Kozen 1994)

$$\models e = f \iff \mathcal{P}(\Sigma^*) \models e = f$$

(Pratt 1980)

$$\mathfrak{R} \models e = f \iff \mathcal{P}(\Sigma^*) \models e = f$$

Main result

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Palka's proof relies on Kozen's completeness theorem. She writes:

> *...an independent proof of [the finite model property] would provide a quite different proof of the Kozen completeness theorem, based on purely logical tools. We defer this task to further research.* (Palka 2005)

We found such a proof — with many ideas inspired by Palka.

Main result

A roadmap

Need to show: if $\mathfrak{F} \models e = f$, then $\models e = f$.

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2. Convert the finite automaton A_e into a finite monoid M_e
3. Translate the finite monoid M_e into a finite KA K_e
4. Prove something about interpretations inside K_e
5. Apply the premise that $\models e = f$

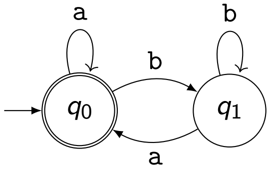
Expressions to automata

Definition

An automaton is a tuple $A = (Q, \rightarrow, I, F)$ where

- Q is a finite set of *states*; and
- $\rightarrow \subseteq Q \times \Sigma \times Q$ is the *transition relation*;
- $I \subseteq Q$ is the set of *initial states*
- $F \subseteq Q$ is the set of *accepting states*

We write $q \xrightarrow{\mathtt{a}} q'$ when $(q, \mathtt{a}, q') \in \rightarrow$.



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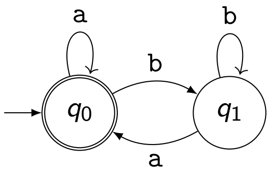

The *language* of $q \in Q$ is $L_A(q) = \{\mathtt{a}_1 \cdots \mathtt{a}_n \in \Sigma^* : q \xrightarrow{\mathtt{a}_1} \circ \cdots \circ \xrightarrow{\mathtt{a}_n} q' \in F\}$

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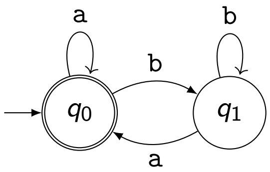

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The language of A is given by $\bigcup_{q \in I} L_A(q)$.

Expressions to automata

Lemma (c.f. Kleene 1956; Brzozowski 1964; Antimirov 1996)

For every e, we can construct an automaton A_e that accepts the language of e.

Automata to monoids

Let $A = (Q, \rightarrow, I, F)$ be an automaton.

Definition (Transition monoid; McNaughton and Papert 1968)

$(M_A, \circ, \Delta)$ is the monoid where $M_A = \{\xrightarrow{\mathtt{a}_1} \circ \cdots \circ \xrightarrow{\mathtt{a}_n} : \mathtt{a}_1 \cdots \mathtt{a}_n \in \Sigma^*\}$.

Monoids to Kleene algebras

Lemma (Palka 2005)

Let $(M, \cdot, 1)$ *be a monoid. Now* $(\mathcal{P}(M), \cup, \otimes, ^{\circledast}, \emptyset, \{1\})$ *is a KA, where*

$$T \otimes U = \{t \cdot u : t \in T \wedge u \in U\} \qquad T^{\circledast} = \{t_1 \cdots t_n : t_1, \ldots, t_n \in T\}$$

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Given an expression e, we can now obtain a *finite* KA $K_e = \mathcal{P}(M_{A_e})$.

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Let $e, f \in \mathrm{Exp}$. *If* $K_e \models e = f$ *and* $K_f \models e = f$, *then* $\models e = f$.

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Lemma

Let $e, f \in \mathrm{Exp}$. *If* $K_e \models e = f$ *and* $K_f \models e = f$, *then* $\models e = f$.

Theorem (Finite model property)

If $\mathfrak{F} \models e = f$ *then* $\models e = f$.

Peeling the onion

Solving automata

Definition

Let $(Q, \rightarrow, I, F)$ be an automaton. A *solution* is a function $s : Q \rightarrow \text{Exp}$ such that

$$\models F(q) + \sum_{q \xrightarrow{\mathtt{a}} q'} \mathtt{a} \cdot s(q') \leq s(q) \qquad\qquad F(q) = \begin{cases} 1 & q \in F \\ 0 & q \notin F \end{cases}$$

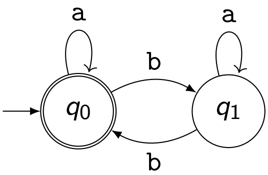
Peeling the onion

Solving automata

Example

For the automaton on the right, a solution satisfies

$$\models 1 + \mathtt{a} \cdot s(q_0) + \mathtt{b} \cdot s(q_1) \leq s(q_0)$$
$$\models 0 + \mathtt{a} \cdot s(q_1) + \mathtt{b} \cdot s(q_0) \leq s(q_1)$$



Peeling the onion

Solving automata

Example (Continued)

We start with the second condition:

$$0 + \mathtt{a} \cdot s(q_1) + \mathtt{b} \cdot s(q_0) \leq s(q_1)$$

Peeling the onion

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which by the fixpoint rule implies

$$\mathtt{a}^* \cdot \mathtt{b} \cdot s(q_0) \leq s(q_1)$$

Peeling the onion

Solving automata

Example (Continued)

Now we look at the second condition

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Substituting $\mathtt{a}^* \cdot \mathtt{b} \cdot s(q_0) \leq s(q_1)$ we get

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which rewrites to

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By the fixpoint rule

$$(\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^* \leq s(q_0)$$

Example (Continued)

We now have two lower bounds:

$$(\mathtt{a} + \mathtt{b} \cdot \mathtt{a}^* \cdot \mathtt{b})^* \leq s(q_0)$$
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It turns these are also solutions to A — thus we found the least solution.

Peeling the onion

Solving automata

Theorem (Kleene 1956; see also Conway 1971)

Every automaton admits a least solution (unique up to equivalence).

Peeling the onion

Solving automata

Theorem (Kleene 1956; see also Conway 1971)

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When A is an automaton, we write

- $\overline{A}(q)$ for the least solution to A at q
- $\lfloor A \rfloor$ for the sum of $\overline{A}(q)$ for $q \in I$

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- $\lfloor A \rfloor$ for the sum of $\overline{A}(q)$ for $q \in I$

Lemma
If $e \in \mathrm{Exp}$, *then* $\models \lfloor A_e \rfloor \leq e$.

Definition (Transition automaton; McNaughton and Papert 1968)

Let $R \in M_A$. We write $A[R]$ for the *transition automaton* $(M_A, \rightarrow_\circ, \{\Delta\}, \{R\})$ where

$$P \xrightarrow{\mathtt{a}}_\circ Q \iff P \circ \xrightarrow{\mathtt{a}} = Q$$

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Intuition: $w \in L(A[R])$ means $q \; R \; q'$ iff w traces from q to q' in A.

Peeling the onion

Solving monoids

Lemma (Solving transition automata)

Let A be an automaton, let $q \in Q$ and let $R \in M_A$ with $q \; R \; q_f \in F$. We have

$$\models \lfloor A[R] \rfloor \leq \overline{A}(q)$$

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Let $h_e : \Sigma \to K_e$ be given by $h_e(\mathtt{a}) = \{\xrightarrow{\mathtt{a}}_e\}$.

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Lemma

Let $e \in \mathrm{Exp}$ and let $R \in \widehat{h_e}(e)$. Then $\models \overline{A_e[R]} \leq e$.

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Solving Kleene algebras

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Let $h_e : \Sigma \to K_e$ be given by $h_e(\mathtt{a}) = \{\xrightarrow{\mathtt{a}}_e\}$.

Lemma

Let $e, f \in \mathrm{Exp}$. We have that

$$\models f \leq \sum_{R \in \widehat{h_e}(f)} \lfloor A_e[R] \rfloor$$

Proof sketch.

By induction on f. □

Lemma

Let $e, f \in \mathrm{Exp}$. If $K_e \models e = f$ and $K_f \models e = f$, then $\models e = f$.

Proof.

Since $K_e \models e = f$, we have that $\widehat{h_e}(e) = \widehat{h_e}(f)$; we can then derive

$$\models f \leq \sum_{R \in \widehat{h_e}(f)} \lfloor A_e[R] \rfloor = \sum_{R \in \widehat{h_e}(e)} \lfloor A_e[R] \rfloor \leq e$$

By a similar argument, $\models e \leq f$; the claim then follows. □

Peeling the onion

The grand finale

Theorem
If $\mathfrak{F} \models e = f$, *then* $\models e = f$.

Proof.
Since K_e and K_f are finite KAs, we have that $K_e \models e = f$ and $K_f \models e = f$.

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The proof then follows by the previous lemma. □

Some thoughts

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  - Upshot: a proof-theoretic result for KA: "right-hand elimination".

Coq formalization

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- Some concepts are encoded differently; ideas remain the same.

Further open questions

- Can we apply these ideas to *guarded Kleene algebra with tests*?

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Bonus: extending the model theory

Lemma
If $\mathfrak{FR} \models e = f$, *then* $\models e = f$.

Bonus: extending the model theory

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If $\mathfrak{FR} \models e = f$*, then* $\models e = f$*.*

Proof sketch.
We show that $\mathfrak{FR} \models e = f$ implies $\mathcal{P}(\Sigma^*) \models e = f$. For $n \in \mathbb{N}$, choose

$$\Sigma_n = \{w \in \Sigma^* : |w| \leq n\} \qquad h_n : \Sigma \to \mathcal{R}(\Sigma_n), \mathtt{a} \mapsto \{(w, w\mathtt{a}) : w\mathtt{a} \in \Sigma_n\}$$

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For all $w \in \Sigma_n$ and regular expressions g, we now have $w \in \widehat{\ell}(g)$ iff $(\epsilon, w) \in \widehat{h_n}(g)$.

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This means that $\mathcal{P}(\Sigma^*), \ell \models e = f$, whence $\mathcal{P}(\Sigma^*) \models e = f$. □

Bonus: pomsets

Expressions in *concurrent KA* (CKA) are generated by

$$e, f ::= 0 \mid 1 \mid \mathtt{a} \in \Sigma \mid e + f \mid e \cdot f \mid e \parallel f \mid e^* \mid e^\dagger$$

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Definition (Bi-KA)

A *bi-KA* is a tuple $(K, +, \cdot, \parallel, {}^*, {}^\dagger, 0, 1)$ where

- $(K, +, \cdot, {}^*)$ and $(K, +, \parallel, {}^\dagger)$ are both KAs, and
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A *weak bi-KA* is a bi-KA without the †.

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Definition (Concurrent KA)

A *(weak) concurrent KA* is a (weak) bi-KA K satisfying

$$(e \parallel g) \cdot (f \parallel h) \leq (e \cdot f) \parallel (g \cdot h)$$

Bonus: pomsets

Example

The *bi-KA of pomset languages* over Σ is $(\mathcal{P}(\mathrm{Pom}(\Sigma)), \cup, \cdot, \parallel, {}^*, {}^\dagger, \emptyset, \{1\})$, where

- $\mathrm{Pom}(\Sigma)$ denotes the set of pomsets over Σ;
- 1 denotes the empty pomset;
- $L \cdot L' = \{U \cdot V : U \in L, V \in L'\}$ and similarly for $\parallel$; and
- $L^* = \{1\} \cup L \cup L \cdot L \cup \cdots$ and $L^\dagger = \{1\} \cup L \cup L \parallel L \cup \cdots$.

Bonus: pomsets

Example

The *concurrent KA of pomset ideals* over Σ is $(\mathcal{I}(\Sigma), \cup, \cdot, \parallel, {}^*, {}^\dagger, \emptyset, \{1\})$, where

- $\mathcal{I}(\Sigma)$ contains the pomset languages downward-closed under $\sqsubseteq$; and
- the operators are as for bi-KA, but followed by downward closure under $\sqsubseteq$.

Bonus: pomsets

Theorem (Laurence and Struth 2014)

Let e and f be (weak) concurrent KA expressions.

Now $\mathcal{P}(\mathrm{Pom}(\Sigma)) \models e = f$ if and only if $K \models e = f$ for all (weak) bi-KAs K

Bonus: pomsets

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Theorem (Laurence and Struth 2017; K., Brunet, Silva, et al. 2018)

Let e and f be weak concurrent KA expressions.

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Bonus: pomsets

Conjecture

Let e and f be concurrent KA expressions.

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Current techniques do not work!

Bonus: pomsets

`<speculation>`

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1. Translate CKA expressions to automata

   ⇒ Pomset automata (K., Brunet, Luttik, et al. 2019)

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3. Translate bimonoids to concurrent KAs.

   ⇒ essentially the same recipe?

Bonus: pomsets

</speculation>

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