Decision problems for Clark-congruential languages

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Introduction

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猫 and 犬 are (almost) *syntactically congruent*:

$$u\text{猫}v \in \text{Japanese} \quad \text{“} \Longleftrightarrow \text{”} \quad u\text{犬}v \in \text{Japanese}$$

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. . . but how to represent the language?

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Definition (Informal)

A grammar is *Clark-congruential* (*CC*) if words derived from the same symbol are syntactically congruent for its language.

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If S derives w and x in G_1, then $uwv \in L$ implies $uxv \in L$ — G_1 is CC.

However: T derives a and ϵ in G_2. Now, $a \in L$ but $\epsilon \notin L$ — G_2 is not CC.

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- Given a grammar H, does $L(G) = L(H)$ hold? If not, give a counterexample.

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Is this decidable?

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Equivalence problem

Given grammars G_1 and G_2, does $L(G_1) = L(G_2)$ hold?

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Given a grammar G, and $w, x \in \Sigma^*$, are w and x syntactically congruent for $L(G)$?

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Equivalence and congruence are undecidable for grammars in general.[2]

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CC languages

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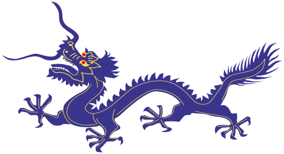
Context-free languages

CC languages



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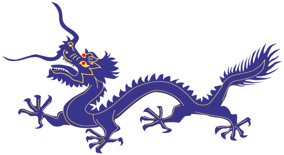

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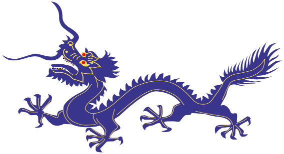

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$$\frac{\forall u, v \in \Sigma^*.\ uwv \in L \iff uxv \in L}{w \equiv_L x}$$

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Definition (More formal)

We say G is CC when for $A \in V$ and $w, x \in L(G, A)$, we have $w \equiv_{L(G)} x$.

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- If w is shorter than x, then $w \preceq x$.
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For $\alpha \in (\Sigma \cup V)^*$ with $L(G, \alpha) \neq \emptyset$, write $\vartheta_G(\alpha)$ for the $\preceq$-minimum of $L(G, \alpha)$.

Deciding congruence

Let G be CC.

We mimic an earlier method to decide congruence.[5]

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Decidable (Sénizergues 1997)

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$L(M_w) = L(M_x)$ *if and only if* $w \equiv_{L(G)} x$.

Theorem

Let $w, x \in \Sigma^*$. *We can decide whether* $w \equiv_{L(G)} x$.

Deciding equivalence

Analogous to a result about NTS grammars,[6] we find

Lemma

Let $G_1 = \langle V_1, \rightarrow_1, I_1 \rangle$ *and* $G_2 = \langle V_2, \rightarrow_2, I_2 \rangle$ *be CC.*

Then $L(G_1) = L(G_2)$ *if and only if*

(i) *for all* $A \in I_1$ *, it holds that* $\vartheta_{G_1}(A) \in L(G_2)$ *(and vice versa)*

(ii) *for all pairs* $u \rightsquigarrow_{G_1} v$ *generating* $\rightsquigarrow_{G_1}$ *, also* $u \equiv_{L(G_2)} v$ *(and vice versa)*

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Theorem

Let G_1 *and* G_2 *be CC. We can decide whether* $L(G_1) = L(G_2)$*.*

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- Adjust learning algorithm to have CC grammars as hypotheses.

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- That is, hypothesis grammars may not be CC!

Two plausible fixes:

- Adjust learning algorithm to have CC grammars as hypotheses.
- Extend decision procedure, requiring only one grammar to be CC.

Further work

Many open questions:

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- Is the language of every CC grammar a DCFL?
- Is it decidable whether a given grammar is CC?

Bonus: grammar to DPDA

Lemma

Let G be CC, and let R be regular.

We can create a CC grammar G_R such that $L(G_R) = L(G) \cap R$.

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Lemma

Let $h : \Sigma^* \to \Sigma^*$ be a strictly alphabetic morphism, that is, $h(a) \in \Sigma$ for all $a \in \Sigma$.

We can create a CC grammar G^h such that $L(G^h) = h^{-1}(L(G))$.

Bonus: grammar to DPDA

For $a \in \Sigma$, add $\bar{a}$ to Σ.

Let $h : \Sigma \to \Sigma$ be such that $h(a) = h(\bar{a}) = a$.

Create G^h such that $L(G^h) = h^{-1}(L(G))$.

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Intuition

G^h is the same as G, but positions in every word can be "marked" by $\bar{\ }$.

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Note that $\mathcal{I}_G$ is a regular language.

Create G_w such that $L(G_w) = L(G^h) \cap \mathcal{I}_G \bar{w} \mathcal{I}_G$.

Now $G_w = \{u\bar{w}v : uwv \in L(G),\ u, v \in \mathcal{I}_G\}$.

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Note that $\mathcal{I}_G$ is a regular language.

Create G_w such that $L(G_w) = L(G^h) \cap \mathcal{I}_G \bar{w} \mathcal{I}_G$.

Now $G_w = \{u\bar{w}v : uwv \in L(G),\ u, v \in \mathcal{I}_G\}$.

Intuition

$L(G_w)$ has words in $L(G)$ with w as a marked substring, with context reduced by $\rightsquigarrow_G$.

Bonus: grammar to DPDA

Lemma

Without loss of generality, every rule generating $\rightsquigarrow_{G_w}$ *overlaps and preserves* $\bar{w}$.

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We can now create a reduction $\rightsquigarrow_{G[w]}$ and a finite set S_w such that

- Every rule generating $\rightsquigarrow_{G[w]}$ contains and preserves $\sharp$.
- $\{x \in \Sigma^* : x \rightsquigarrow_{G[w]} y \in S_w\} = \{u\sharp v : uwv \in L(G),\ u, v \in \mathcal{I}_G\}$

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The DPDA M_w acts by reading $u\sharp v$ up to $\sharp$, putting the input on the stack. Then:

- Pop from the stack or read from input into two buffers (encoded in state).
- Whenever possible, reduce according to the rules from $\rightsquigarrow_{G[w]}$.

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- Pop from the stack or read from input into two buffers (encoded in state).
- Whenever possible, reduce according to the rules from $\rightsquigarrow_{G[w]}$.
- When the buffer resembles S_w and the input and stack are empty, accept.

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- Whenever possible, reduce according to the rules from $\rightsquigarrow_{G[w]}$.
- When the buffer resembles S_w and the input and stack are empty, accept.

With some analysis, we find that $L(M_w) = \{u\sharp v : uwv \in L(G),\ u, v \in \mathcal{I}_G\}$.

Bonus: deciding Clark-congruentiality

Given a congruence $\equiv$, we can extend it a congruence $\hat{\equiv}$ on $(\Sigma \cup V)^*$, by stipulating

$$\frac{\vartheta_G(\alpha) \equiv \vartheta_G(\beta)}{\alpha \mathrel{\hat{\equiv}} \beta}$$

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Lemma

Let $\equiv$ be a congruence on Σ^*.

The following are equivalent:

(i) *For all productions $A \to \alpha$, it holds that $A \mathrel{\hat{\equiv}} \alpha$*

(ii) *For all $A \in V$ and $w, x \in L(G, A)$, it holds that $w \equiv x$.*

Bonus: deciding Clark-congruentiality

Theorem

If $\equiv_{L(G)}$ is decidable, then we can decide whether G is CC.

Proof.

For $A \to \alpha$, check whether $A \mathrel{\hat{\equiv}}_{L(G)} \alpha$, i.e., whether $\vartheta_G(A) \equiv_{L(G)} \vartheta_G(\alpha)$. □

Bonus: deciding Clark-congruentiality

Theorem

If $\equiv_{L(G)}$ is decidable, then we can decide whether G is CC.

Proof.

For $A \to \alpha$, check whether $A \triangleq_{L(G)} \alpha$, i.e., whether $\vartheta_G(A) \equiv_{L(G)} \vartheta_G(\alpha)$. □

Corollary

If $L(G)$ is a deterministic CFL, then it is decidable whether G is CC.