Concurrent Kleene Algebra: Free Model and Completeness

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Introduction

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Are these programs equivalent?

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Kleene Algebra (KA) provides an *algebraic framework* to do this.

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repetition	e^*

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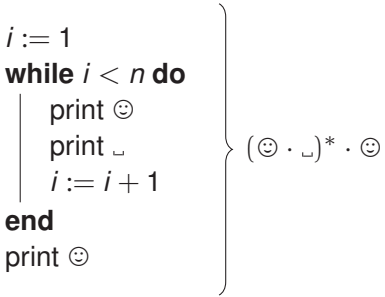

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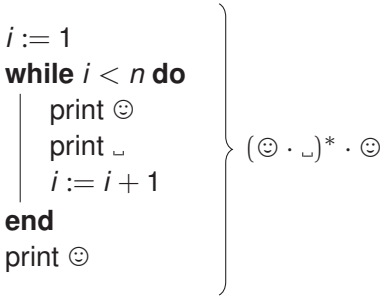

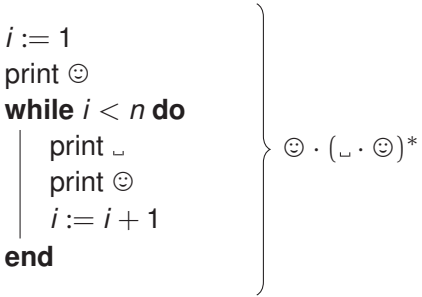

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Axioms of KA:

$$e + 0 \equiv e \qquad e + e \equiv e \qquad e + f \equiv f + e \qquad e + (f + g) \equiv (e + f) + g$$

$$e \cdot 0 \equiv 0 \equiv 0 \cdot e \qquad e \cdot 1 \equiv e \equiv 1 \cdot e \qquad e \cdot (f \cdot g) \equiv (e \cdot f) \cdot g$$

$$e \cdot (f + g) \equiv e \cdot f + e \cdot g \qquad (e + f) \cdot g \equiv e \cdot g + f \cdot g$$

$$1 + e \cdot e^* \equiv e^* \qquad e \cdot f + g \leqq f \implies e^* \cdot g \leqq f$$

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$$☺ \cdot (␣ \cdot ☺)^* \equiv (☺ \cdot ␣)^* \cdot ☺$$

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Theorem (Kozen 1990)

The axioms for KA *are sound &* ***complete*** *for equivalence:*

$$e \equiv f \iff \mathcal{L}(e) = \mathcal{L}(f)$$

$\mathcal{L}(e)$ *is the regular language interpretation of* e*.*

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Upshot:

- to check KA equivalence is to check regular language equivalence
- through Kleene's theorem, this means checking DFA equivalence
- sophisticated (near-linear) algorithms exist to do this

Adding concurrency

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$$e \parallel 1 \equiv e \qquad\qquad e \parallel 0 \equiv 0$$

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$$e \parallel f \equiv f \parallel e \qquad e \parallel (f \parallel g) \equiv (e \parallel f) \parallel g$$

$$e \parallel 1 \equiv e \qquad e \parallel 0 \equiv 0 \qquad e \parallel (f + g) \equiv e \parallel f + e \parallel g$$

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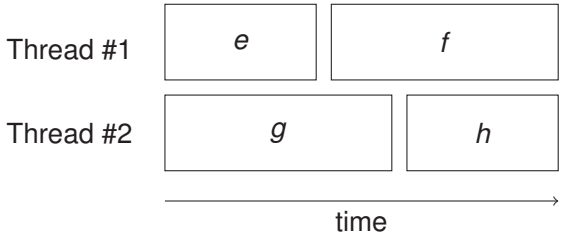

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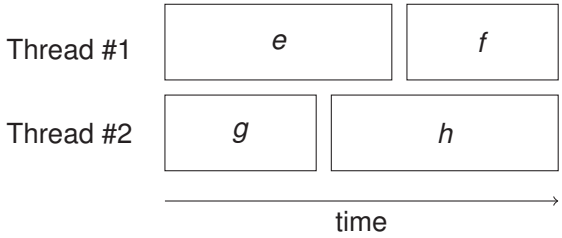

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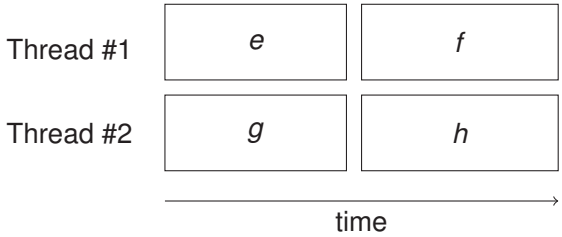

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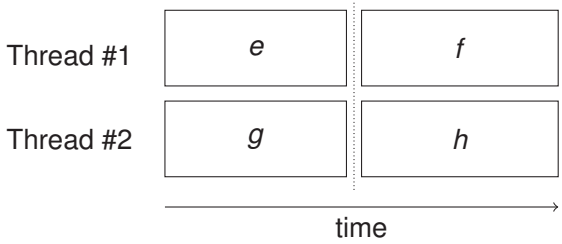

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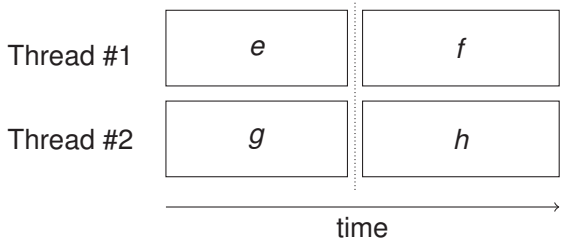

Equationally: $(e \parallel g) \cdot (f \parallel h) \leqq (e \cdot f) \parallel (g \cdot h)$.

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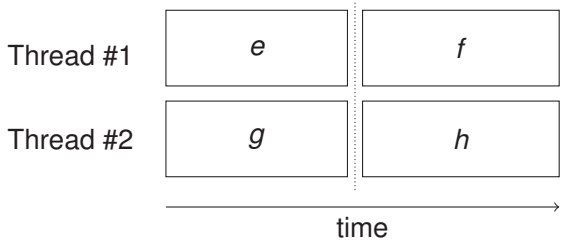

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$$p \leqq q \iff p + q \equiv q$$

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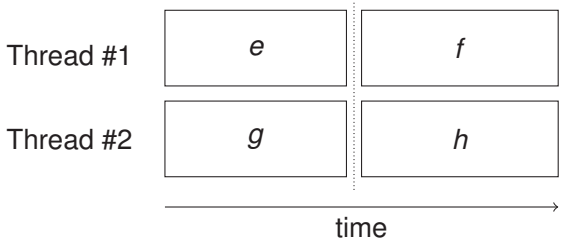

Equationally: $(e \parallel g) \cdot (f \parallel h) \leqq (e \cdot f) \parallel (g \cdot h)$.

Nondeterministic interleaving as special case: $e \cdot f + f \cdot e \leqq e \parallel f$.

Adding concurrency

Question

Can we have a regular interpretation $[\![-]\!]$ *such that* $e \equiv f \iff [\![e]\!] = [\![f]\!]$*?*

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NB: $[\![-]\!]$ should generalize $\mathcal{L}(-)$: for $\|$-less terms, $\mathcal{L}(e)$ should resemble $[\![e]\!]$.

Regular interpretation: first attempt

Partially ordered multiset (pomset):

$$a \cdot b \cong a \longrightarrow b$$

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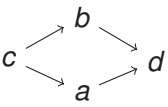

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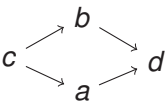

Composition lifts to *sets of pomsets* in the obvious way.

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Straightforward semantics: $(\!|-|\!) : \mathcal{T} \to 2^{\text{Pomsets}}$ given by

$$(\!|0|\!) = \emptyset \qquad (\!|e + f|\!) = (\!|e|\!) \cup (\!|f|\!) \qquad (\!|e^*|\!) = (\!|e|\!)^*$$

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Problem: $(\!|-|\!)$ is not sound for the exchange law.

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Problem: $(\!|-|\!)$ is not sound for the exchange law.

For instance: $a \cdot b \leqq a \parallel b$ should imply that $(\!|a \cdot b|\!) \subseteq (\!|a \parallel b|\!)$, but

$$(\!|a \cdot b|\!) = \left\{ \; a \longrightarrow b \; \right\} \qquad (\!|a \parallel b|\!) = \left\{ \; a \qquad b \; \right\}$$

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Axioms to build $\approx$ are axioms for $\equiv$, minus exchange law.

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Theorem (Laurence and Struth 2014)

The axioms for $\approx$ are sound & complete w.r.t. $(\!|-|\!)$:

$$e \approx f \iff (\!|e|\!) = (\!|f|\!)$$

Regular interpretation: second attempt

We define the *subsumption order* $\sqsubseteq$ on pomsets.

Intuition: $U \sqsubseteq V$ if

i. U and V have the same events, and
ii. U has all order in V (and possibly more)

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$$a \longrightarrow b \sqsubseteq a \quad b$$

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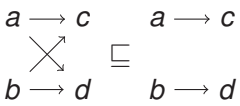

Regular interpretation: second attempt

"Fixed" semantics: $[\![e]\!] = (\!|\, e \,|\!)\downarrow$.

downward closure w.r.t. $\sqsubseteq$

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Previous problem no longer occurs:

$$[\![a \cdot b]\!] = \left\{ \; a \to b \; \right\} \subseteq \left\{ \; a \to b \;,\; a \leftarrow b \;,\; a \quad b \; \right\} = [\![a \parallel b]\!]$$

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Lemma (Hoare et al. 2009)

The axioms for $\equiv$ are sound w.r.t. $[\![-]\!]$, i.e., $e \equiv f$ implies $[\![e]\!] = [\![f]\!]$.

Closure

Definition

Let $e \in \mathcal{T}$; a *closure* of e is a term $e{\downarrow}$ such that

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If $[\![e]\!] = [\![f]\!]$, then $(\!| e{\downarrow} |\!) = (\!| f{\downarrow} |\!)$, thus $e{\downarrow} \approx f{\downarrow}$. Therefore, $e \equiv e{\downarrow} \equiv f{\downarrow} \equiv f$. □

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*If $e \in \mathcal{T}$, then we can **compute** a term $e{\downarrow}$ that is a closure of e.*

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The axioms for CKA *are sound & complete w.r.t. $[\![-]\!]$:*

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The latter can be decided; c.f. [Brunet, Pous, and Struth 2017].

Further work

- Explore coalgebraic perspective:
  - Efficient equivalence checking through bisimulation?
  - Can completeness be shown coalgebraically?
- Add “parallel star” operator — closure method does not apply.
- Extend *Kleene Algebra with Tests* (KAT) to add concurrency.
- Extend extend NetKAT with concurrency.

Thank you for your attention

Implementation: https://doi.org/10.5281/zenodo.926651.

Extended paper: https://arxiv.org/abs/1710.02787.